QUANTITATIVE FINANCE

School of Finance, SUFE

Exercise for Part II (Spring, 2020)

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1. There are three assets with the following covariance matrix $\boldsymbol{V}$ and expected returns $\bar{\boldsymbol{r}}$.

$$\boldsymbol{V} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \quad \bar{\boldsymbol{r}} = \begin{bmatrix} 0.4 \\ 0.6 \\ 0.6 \end{bmatrix}.$$

   a. Find the Global Minimum Variance Portfolio.

   b. Given the portfolio expected return r_p, find the corresponding minimum-variance portfolio.

2. In the Markowitz Model, the efficient portfolio with minimum variance is called the Global Minimum Variance Portfolio (GMVP). For the GMVP with return r_G and any portfolio with return r_P, prove that $Cov(r_P, r_G) = Var(r_G)$, where $Cov(\cdot,\cdot)$ and $Var(\cdot)$ denote covariance and variance, respectively.

3. (Dual Problem) In the Markowitz Model, we minimize the portfolio variance for any given level of portfolio expected return. Actually, we can also maximize the expected return of a portfolio for a given variance level (e.g., $\bar{\sigma}_p^2$) of the portfolio. Write down the maximization problem.

4. (Transaction Costs) The Markowitz Model assumes no transaction costs. Now we extend the model to include transaction costs.

   a. Transaction costs for asset i are proportional to w_i and thus can be written as $c_i w_i$, where c_i is a constant. What is the variance minimization problem for this case?

   b. Consider a more realistic case. Transaction costs for asset i are proportional to $|w_i|$ and different for opposite directions of trades. E.g., for buy-in, transaction costs are $c_B w_i$, and for sell-out, they are $-c_S w_i$. What is the variance minimization problem for this case? (Hint: you may use $\max(0, w_i)$ in the constraints.)

5. (Tracking) Suppose that it is impractical to use all the assets that are incorporated into a specified portfolio (such as a given portfolio). One alternative is to find the portfolio, made up of a given set of n stocks, that tracks the specified portfolio most closely—in the sense of minimizing the variance of the difference in returns.

   Specifically, suppose that the target portfolio has (random) rate of return r_M. Suppose that there are n assets with (random) rates of returns $r_1, r_2, ..., r_n$. We wish to find the portfolio rate of return

   $$r = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n,$$

   (with $\sum_{i=1}^{n} w_i = 1$) minimizing $Var(r - r_M)$.

   a. Find a set of F.O.C.s for the w_i's.

   b. Although this portfolio tracks the desired portfolio most closely in terms of variance, it may sacrifice the mean. Hence a logical approach is to minimizing the variance of the tracking errors subject to achieving a given mean return. As the mean is varied, this results in a family of portfolios that are efficient in a new sense—say, tracking efficient. Find the set of F.O.C.s for the w_i's that are tracking efficient.